10 Summary: Review of two Cauchy problems and some practice problems

10.1 Comparison of the two second-order Cauchy problems

1D Heat equation	**1D Wave equation**
$u_t = Du_{xx} + \mathbb{S}(x,t) \quad \|x\| < \infty, t > 0$	$u_{tt} = c^2 u_{xx} + \mathbb{S}(x,t) \quad \|x\| < \infty, t > 0$
homogeneous case: $\mathbb{S} \equiv 0$	
General solution in this case:	
$u(x,t) = C_1 \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-s^2} ds + C_2$	$u(x,t) = F(x - ct) + G(x + ct)$
Cauchy Problem: $\|x\| < \infty, t > 0$	
$u_t = Du_{xx} + \mathbb{S}(x,t) \quad \|x\| < \infty, t > 0$ $u(x,0) = f(x), \quad \|x\| < \infty$	$u_{tt} = c^2 u_{xx} + \mathbb{S}(x,t) \quad \|x\| < \infty, t > 0$ $u(x,0) = f(x), u_t(x,0) = g(x), \quad \|x\| < \infty$
$u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4Dt} f(y) dy +$ $\frac{1}{2\sqrt{\pi D(t-\tau)}} \int_0^t \int_{-\infty}^{\infty} e^{-(x-y)^2/4D(t-\tau)} \mathbb{S}(y,\tau) dy d\tau$	$u(x,t) = \frac{1}{2}\{f(x-ct) + f(x+ct)\} +$ $\frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy + \frac{1}{2c} \int_0^t \int_{\Delta(x,t)} \mathbb{S}(y,\tau) dy d\tau$
Properties to remember: solutions are infinitely differentiable	solutions are no more smooth than the i.c.s
infinite speed of propagation	finite speed of propagation (c)
no real characteristics	2 families of real characteristics
information from initial data gradually lost	information from initial data transported indefinitely along characteristics

Recall that we previously made a partial comparison of the solution to the wave and heat equations by considering the two problems

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty,\ t > 0 \\ &u(x,0) = H(1-|x|),\ \frac{\partial u}{\partial t}(x,0) = 0, \quad |x| < \infty \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty,\ t > 0 \\ &u(x,0) = H(1-|x|), \quad |x| < \infty \end{aligned} \tag{2}$$

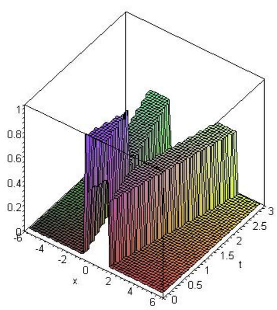

Figure 1: Graph of solution to problem 1

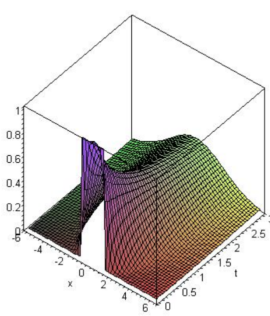

Figure 2: Graph of solution to problem 1

Figure 1 shows the propagation of the singularities along the real characteristics, and also shows the nature of Huygen's principle and conservation of energy principle, while Figure 2 shows the forgetfulness of the solution to the heat equation.

10.2 Extra practice problems

1. (a) Consider the wave equation $u_{tt} = u_{xx}$ in $\mathbb{R} \times \mathbb{R}^+$, with generic initial data $u(x,0) = f(x),\ u_t(x,0) = g(x)$. Write down d'Alembert's formula for the solution to the Cauchy problem, and for any given (x_0, t_0) in the domain, draw a characteristic triangle associated with the point (x_0, t_0). Label everything in the graph, including the equation for the characteristics through the point, and the domain of dependence on the x-axis.

   (b) Now define $f(x) = \sin(2x), g(x) = e^{-|x|}$ and compute the solution $u(x,t)$.

2. Consider the equation $u_{tt} + e^{-t}u_t = u_{xx} + sinh(x)u_x$. Is the equation hyperbolic, parabolic, or elliptic? Why?

3. Given the vibrating string equation $u_{tt} = u_{xx}$ in $\mathbb{R} \times \mathbb{R}^+$, what is the definition of domain of dependence for a point (x_0, t_0)? What is the domain of dependence for a point $(x,t) = (4,3)$? What is the region of influence of the point $(x,t) = (3,0)$?

4. For the problem $u_{tt} = 4u_{xx}$ in $\mathbb{R} \times \mathbb{R}^+$, $u(x,0) = 2H(x)$, $u_t(x,0) = H(x+1)$, construct the appropriate solution representation.

5. What is the general solution of $3u_y + u_{xy} = 0$ in terms of two general functions?
   (Ans: $u = f(y)e^{-3x} + g(x)$)

6. Consider the problem $u_t = D_0 e^{-at} u_{xx}$ in $\mathbb{R} \times \mathbb{R}^+$, with $u(x,0) = f(x)$. (So we have a process where the diffusivity is slowly decreasing, $a > 0$ small.) Write the representation of the solution.
   (Hint: let $u(x,t) = v(x,\tau)$, where $\tau := \int_0^t D_0 e^{-as} ds$, so $\frac{\partial}{\partial t} = D_0 e^{-at} \frac{\partial}{\partial \tau}$.)

7. Write the solution formula for the problem $u_t = u_{xx} - 5$, $u(x,0) = f(x)$.

8. For the linear cable equation for a nerve dendrite, $Cv_t + \frac{v-E}{R} = \frac{a}{2R_i} v_{xx}$, let $v(x,0) = v_0(x)$, and reduce the equation down to a heat equation for w via the transformation $v(x,t) = E + e^{-t/RC} w(x,t)$, and solve the w equation problem. Then write the solution for $v(x,t)$.

9. What type equation is $u_{xx} + 4u_{xy} + 3u_{yy} + 3u_x - u_y + 2u = 0$?

10. Find the characteristics of the equation $u_t + xu_x = 0$.
(Ans: $t = \ln|x| + K$)

11. Solve $u_t + u_x = u^2$, $u(-x, x) = x \in \mathbb{R}$.
(Ans: $u(x,t) = \frac{2t-2x}{4-t^2+x^2}$)

12. Solve $xu_x + yu_y = 1$, $u(1, y) = \ln(y)$, $y > 0$.

13. For the Cauchy problem $u_{tt} = 0.01u_{xx}$, $u(x, 0) = e^{-x^2}$, $u_t(x, 0) = 0$, where is the peak of the waves when $t = 10$?
(Ans: At $x = \pm 1$.)

14. In the specified domain determine the type of equation (elliptic, hyperbolic, parabolic). If the equation changes type in the domain, explain where it is which type.

    (a) $u_{xx} - 6u_{xy} + 12u_{yy} - u_y = x + y$ in the plane.

    (b) $u_{xx} + 2yu_{xy} + xu_{yy} - u_x + 8u = 0$ for $x, y > 0$.

    (Ans: (a) elliptic everywhere; (b) elliptic (resp. hyperbolic) below (above) curve $x - y^2 = 0$, and parabolic on the curve)